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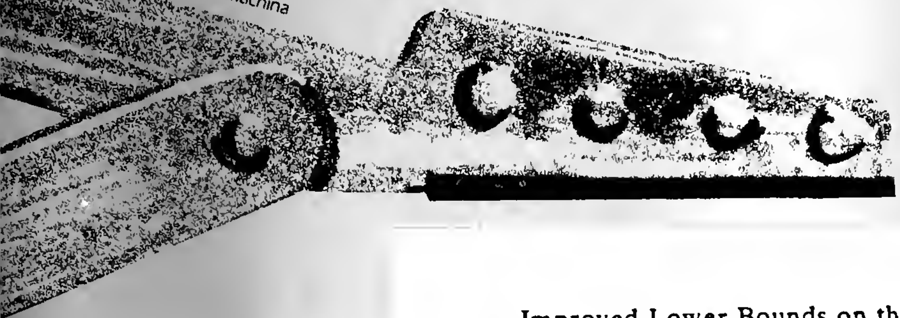

Improved Lower Bounds on the Length of Davenport-Schinzel Sequences

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Micha Sharir

Technical Report No. 204
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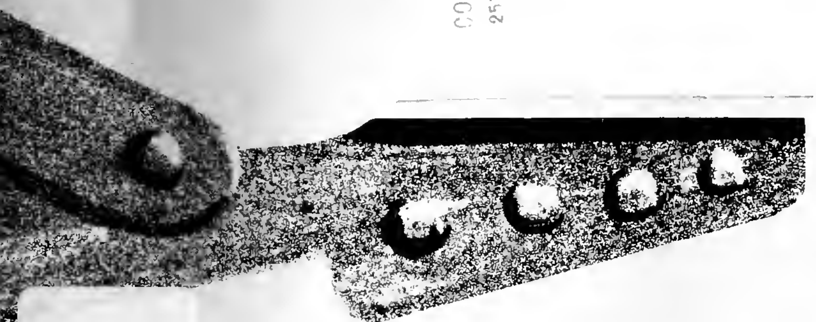

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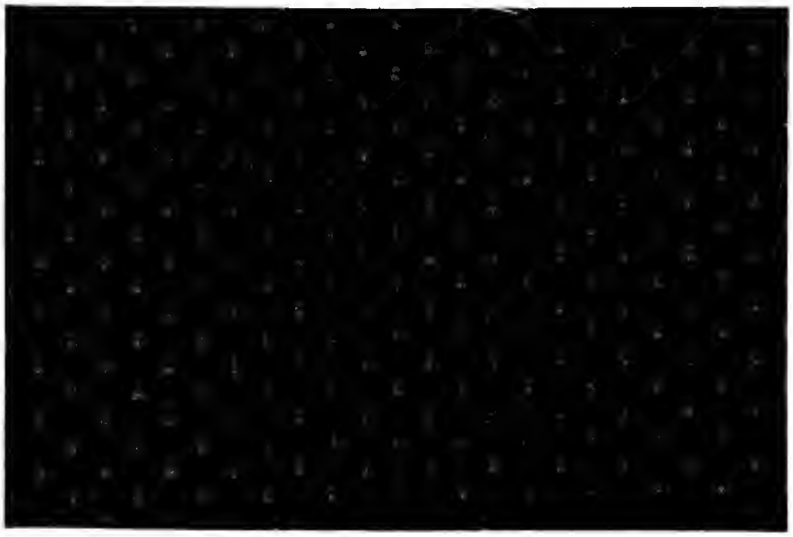

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ABSTRACT

We derive lower bounds on the maximal length $\lambda_s(n)$ of (n,s) Davenport Schinzel sequences. These bounds have the form $\lambda_{2s+1}(n) = \Omega(n\alpha^s(n))$, where $\alpha(n)$ is the extremely slowly growing functional inverse of the Ackermann function. These bounds extend the nonlinear lower bound $\lambda_3(n) = \Omega(n\alpha(n))$ due to Hart and Sharir [HS], and are obtained by an inductive construction based upon the construction given in [HS].

1. Introduction

In this paper we obtain improved nonlinear lower bounds on the maximal length $\lambda_s(n)$ of an (n,s) *Davenport-Schinzel sequence* (a $DS(n,s)$ sequence in short). Such a sequence $U = (u_1, \ldots, u_m)$ is defined to be a sequence composed of n distinct symbols which satisfies the following two conditions:

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(i) $u_i \neq u_{i+1}$ for all i;

(ii) U does not contain any alternating subsequence of two (distinct) symbols whose length is $s+2$; that is, there do not exist $s+2$ indices $i_1 < i_2 < \cdots < i_{s+2}$ such that

$$u_{i_1} = u_{i_3} = u_{i_5} = \cdots = a ,$$
$$u_{i_2} = u_{i_4} = u_{i_6} = \cdots = b ,$$

and $a \neq b$.

These sequences provide a combinatorial characterization of the lower envelope of n continuous functions, each pair of which intersect in at most s points. In fact $\lambda_s(n)$ is equal to the maximal number of connected graph portions of such functions which compose their lower envelope (cf. [DS], [Da], [At]). Analysis of these sequences has thus numerous applications in combinatorial and computational geometry, for which see [At], [HS], [OSY], [BS], [LS].

The major known estimates for $\lambda_s(n)$ are:

(I) $\lambda_1(n) = n$; $\lambda_2(n) = 2n - 1$ (trivial; cf. e.g. [At]).

(II) $\lambda_3(n) = \Theta(n\alpha(n))$ (Hart and Sharir [HS]), where $\alpha(n)$ is the functional inverse of Ackermann's function.

(III) $\lambda_s(n) = O(n\alpha(n)^{O(\alpha(n)^{s-3})})$ (Sharir [Sh]). This is an improvement of the upper bound $\lambda_s(n) \leq C_s n \log^* n$ obtained by Szemeredi [Sz], where C_s is a positive constant depending on s, and where $\log^* n$ is the height of the smallest exponential tower $2^{2^{\cdot^{\cdot^{2}}}}$ which is $\geq n$.

The only previously known nontrivial lower bound on $\lambda_s(n)$ was that obtained in [HS]. In this paper we extend the technique of [HS] to obtain the lower bounds

$$\lambda_{2s+1}(n) = \Omega(n\alpha^s(n))$$

for each $s \geq 1$ and each $n \geq 1$, where the constant of proportionality depends on s.

2. A modified inductive construction of nonlinear *DS* sequences.

Our construction of $DS(n,2s+1)$ sequences of length $\Omega(n\alpha^s(n))$ proceeds by induction on s based on the construction of nonlinear $DS(n,3)$ sequences given in [HS, Section 6]. We assume here familiarity of the reader with the (fairly involved) construction of [HS], but we begin by briefly recalling some of its important features.

Given a tree T, a *generalized path compression* (GPC in short) f on T is a sequence $(x_1, \ldots, x_k)$ of nodes of T, all lying along the same path of T from some leaf to the root, and ordered so that for each i the node x_i is a (not necessarily direct) descendant of x_{i+1}; the compression f amounts to modifying T so that each of $x_1, \ldots, x_{k-1}$ becomes a direct child of x_k, and all other parent-child relationships in T remain unchanged. f is said to have length $k-1$, and to *pass through* the nodes $x_1, \ldots, x_{k-1}$; x_1 is called the *starting node* of f. An (admissible) *path compression scheme* F on T is a sequence of GPC's $(f_1, \ldots, f_t)$ on T such that (i) the starting nodes of the f_i's are arranged in tree postorder, and (ii) the nodes of f_i still lie along a single path of the tree after it has been compressed by $f_1, \ldots, f_{i-1}$ in this

order.

The construction of [HS] yields, for each pair of integers k, m, an admissible path compression scheme on a balanced mostly binary tree $T(k,m)$ such that exactly one generalized path compression (GPC) starts at each leaf of $T(k,m)$, and such that each of these GPC's has length k. The tree $T(k,m)$ is obtained by taking a complete binary tree of depth $B_k(m)$, and by replacing each of its leaves by a copy of a *base tree* $T = T(1,m)$ consisting of a root and m leaves, all of which are direct children of the root. The functions $B_k(m)$ are a collection of rapidly growing functions defined inductively as follows:

$$
\begin{aligned}
&B_1(m) = 0\ , && m \geq 0 \\
&B_k(0) = 1\ , && k \geq 2\ , \\
&B_k(m) = B_k(m-1) + B_{k-1}(2^{B_k(m-1)})\ , && k \geq 2\ ,\ m \geq 1\ .
\end{aligned}
$$

Recall also that in transforming such a path compression scheme to a *DS* sequence $S(k,m)$, the nodes of $T(k,m)$ become *chains* in S (ordered in their tree postorder), and the GPC's become the symbols of which $S(k,m)$ is composed. Each chain c consists of all the symbols (GPC's) which pass through the corresponding tree node n (so n is not the last node of any of them), arranged in *decreasing* postorder of their starting leaves, and $S(k,m)$ is the concatenation of all these chains (in their postorder). Thus $S(k,m)$ is a sequence composed of

$$N(k,m) = m \cdot 2^{B_k(m)} \tag{1}$$

symbols spread over about that many chains. (Note that $S(k,m)$ may not be a proper *DS* sequence, because it can contain pairs of equal adjacent elements.

However, as noted in [HS], the transformation from this sequence to a proper DS sequence is straightforward and does not change the length of the sequence by more than an amount linear in N.)

In what follows it will be convenient to use the functions

$$C_k(m) = 2^{B_k(m)}$$

which thus satisfy the formulas

$$\begin{aligned} &C_1(m) = 1 , && m \geq 0 , \\ &C_k(0) = 2 , && k \geq 2 , \\ &C_k(m) = C_k(m-1)\cdot C_{k-1}(C_k(m-1)) , && k \geq 2,\ m \geq 1 . \end{aligned}$$

In particular note that $N(k,m) = mC_k(m)$. Note also that $C_1(1) = 1$, $C_2(2) = 2$, $C_3(3) = 16$, and $C_4(4) > 2^{2^{2^{21}}}$, as can be easily checked.

Let γ_k be the functional inverse of C_k, i.e.

$$\gamma_k(m) = \min\,\{j : C_k(j) \geq m\}$$

Similarly, let γ be the functional inverse of the diagonalization $C(k) = C_k(k)$. The analysis in Section 6 of [HS] implies

$$A_{k-1}(m) \leq C_k(m) \leq A_k(m+3)$$

for $k \geq 4$, $m \geq 1$. where $A_k(m)$ are the standard Ackermann's functions (cf. [HS] for more detail). These inequalities easily imply (for small values of m this should be checked directly)

$$\gamma(m) - 2 \leq \alpha(m) \leq \gamma(m) + 3 \tag{2}$$

so that these two functional inverses are almost equal. In particular the results of [HS] imply

$$\lambda(m) \geq cm\gamma(m)$$

for some positive constant c and each $m \geq 1$.

Next let $s = 2q+1$ for some $q > 1$. Transform $S(k,m)$ into another longer sequence $U_s(k,m)$ as follows. Take each chain c in $S(k,m)$; suppose c consists of t (distinct) symbols. Then replace c by a maximal $DS(t,s-2)$ sequence c^* composed from the same t symbols so that the leftmost appearances of these symbols in c^* occur in the same order as their single appearances in c. $U_s(k,m)$ is then defined to be the concatenation of all these expanded chains.

Lemma 1: After erasures of adjacent equal elements, $U_s(k,m)$ becomes a $DS(n,s)$ sequence, where $n = N(k,m)$.

Proof: By double induction on k, m. First observe that if for some k, m the sequence $U_s(k,m)$ contains an alternation $a\ b\ a\ \cdots\ b\ a$ of length $s+2 > 5$, then there must exist a chain in $S(k,m)$ which contains both a and b.

Now consider first the case $k = 1$. $T(1,m)$ is just a depth-1 tree consisting of a root and m leaves, and only the leaves become nonempty chains of $S(1,m)$. Moreover, each chain c corresponding to some leaf l contains only one GPC (namely the GPC starting at l), so it has length 1, and thus $c^* = c$. Hence $U_s(1,m) = S(1,m)$ so the claim is obvious in this case.

Consider next the case $k > 1$, $m = 1$, and suppose that the claim is true for all $k' < k$, $m' \geq 1$. It is easy to see that, after erasures, $U_s(k,1)$ becomes equal to $U_s(k-1,2)$, so the claim follows in this case by induction hypothesis for $k-1$.

Finally consider the case $k > 1$, $m > 1$, and suppose the claim is true for all $k' < k$, $m' \geq 1$, and for $k' = k$, $m' < m$. The inductive step in [HS] first

constructs $T(k,m-1)$ and an associated path compression scheme on it by temporarily ignoring the last leaf in each base-tree T. Then it constructs $T(k-1,C_k(m-1))$ and an associated compression scheme by using as a base tree the tree T^* obtained from $T(k,m-1)$ by discarding all nodes other than its root and the $C_k(m-1)$ roots of the copies of the base tree T in $T(k,m-1)$. Finally, the tree $T(k-1,C_k(m-1))$ and the $C_{k-1}(C_k(m-1))$ copies of $T(k,m-1)$, together with their associated compression schemes, are properly merged together in the manner described in [HS] to yield a compression scheme on a tree $T(k,m)$ having the desired properties.

(More specifically, this merging is performed as follows. $T(k,m)$ is obtained from $T(k-1,C_k(m-1))$ by replacing each copy of its base tree T^* by a copy of $T(k,m-1)$ in which each base tree T has now m leaves. The combined path compression scheme on $T(k,m)$ is defined as follows. For a leaf l which is not the last (i.e. the m-th) leaf of its base tree, use the GPC yielded by the compression scheme associated with the copy of $T(k,m-1)$ containing l. If l is the m-th leaf in its base tree, let f_0 be the GPC starting at the root r of that copy of T, as yielded by the compression scheme associated with $T(k-1,C_k(m-1))$; then define the GPC starting at l to be the concatenation of l with f_0. It is shown in [HS] that this compression scheme is admissible, and clearly each of its GPC's has length k.)

Suppose $U_s(k,m)$ contains an alternation $a\ b\ a\ \cdots\ b\ a$ of (odd) length $s+2$. Suppose first that the GPC's a and b both belong to compression schemes on the "small" trees $T(k,m-1)$. If their schemes are associated with different copies of $T(k,m-1)$ then a and b clearly cannot alternate at all in

$S(k,m)$, and thus also in $U_s(k,m)$. If a and b belong to the scheme associated with the same copy of $T(k,m-1)$, then we obtain a contradiction by induction hypothesis for $k,m-1$. Suppose next that a, b both belong to the scheme associated with the "big" tree $T(k-1,C_k(m-1))$. It is easy to check that in this case the induction hypothesis for $k-1$ rules out such an alternation. Suppose finally that a belongs to a scheme associated with a copy of $T(k,m-1)$ and that b belongs to the scheme associated with $T(k-1,C_k(m-1))$. Then, as observed above, a and b must both pass through the same node of $T(k,m)$. But the only nodes through which both types of GPC can pass are the roots of the base-tree T. Thus a and b are GPC's starting at two leaves of the same copy of T, with b starting at the last (rightmost) leaf in that copy. Let r be (the chain in $S(k,m)$ corresponding to) the root of T. Note that b must be the first element in r, and that, after erasures, its first appearance in r^* must also be its first appearance in U_s (indeed, before erasures, the first appearance of b in U_s is in the singleton chain corresponding to b's starting leaf, which immediately precedes r in postorder; thus after erasures this first appearance is erased, making the above claim obvious). It is also easy to check that U_s contains one occurrence of a before r^* (namely at the chain corresponding to the starting leaf of a), and that, following r^*, all remaining occurrences of a in U_s must precede all remaining occurrences of b. These facts, together with the fact that r^* is a $DS(m,s-2)$ sequence and $s-2$ is odd, easily yield the desired contradiction.

□

Let us, in a slight abuse of notation, denote by $U_s(k,m)$ also the length of that sequence (before erasures). Then we have

Lemma 2:

$$U_s(1,m) = m \ , \qquad m \geq 1 \ , \tag{3}$$
$$U_s(k,1) = U_s(k-1,2) + C_k(1) \ , \qquad k \geq 2 \ , \tag{4}$$
$$U_s(k,m) = C_{k-1}(C_k(m-1)) \cdot U_s(k,m-1) + \tag{5}$$
$$C_k(m)(\lambda_{s-2}(m) - \lambda_{s-2}(m-1)) + U_s(k-1,C_k(m-1)) \ , \quad k \geq 2, \ m \geq 2 \ .$$

Proof: (3) follows from the fact that $S(1,m)$ consists of m chains each composed of a single symbol. Thus in this case $U_s(1,m) = S(1,m)$. (4) follows from the construction, noting that the additional chains in $S(k,1)$ which are not present in $S(k-1,2)$ correspond to the leaves of $T(k,1)$, and are thus each composed of a single symbol. Since the number of such leaves is $C_k(1)$, (4) follows. As for (5), note that the first term on its right-hand side represents the sum of the lengths of all the "small" sequences obtained from the compression schemes on the $C_{k-1}(C_k(m-1))$ copies of $T(k,m-1)$ in $T(k,m)$. The third term represents the length of the sequence obtained from the scheme associated with $T(k-1,C_k(m-1))$, in which we can regard each GPC as starting at the m-th leaf of the corresponding base tree and skipping the root of that tree, obtaining this way the full length of the contribution of $S(k-1,C_k(m-1))$ to $S(k,m)$ (and also the full contribution of $U_s(k-1,C_k(m-1))$ to $U_s(k,m)$), but still missing one occurrence of each such GPC f in the resulting sequence $S(k,m)$, namely its occurrence in the chain corresponding to the root of the base tree containing the starting leaf of f. The second term in the right-hand side of c represents the length of all expanded chains of $U_s(k,m)$ containing GPC's from both a scheme associated

with some $T(k,m-1)$ and the scheme associated with $T(k-1,C_k(m-1))$. Indeed, as noted in the proof of Lemma 1, these "common" chains are precisely those corresponding to the roots of the copies of the base tree. Each such root induces in $T(k,m-1)$ a chain consisting of $m-1$ symbols, so the length of this chain in $U_s(k,m-1)$ is $\lambda_{s-2}(m-1)$. However in $T(k,m)$ this root induces a chain consisting of m symbols, so its new length (in $U_s(k,m)$) is $\lambda_{s-2}(m)$. These observations plainly imply (5). □

We can simplify the above recurrence formulas by putting $U_s(k,m) = C_k(m) \cdot Z_k^{(s)}(m)$. Then we have

$$Z_1^{(s)}(m) = m\ , \qquad m \geq 1\ . \tag{6}$$

$$Z_k^{(s)}(1) = \frac{1}{2} Z_{k-1}^{(s)}(2) + 1\ , \qquad k \geq 2\ , \tag{7}$$

$$Z_k^{(s)}(m) = Z_k^{(s)}(m-1) + \lambda_{s-2}(m) - \lambda_{s-2}(m-1) + \frac{Z_{k-1}^{(s)}(C_k(m-1))}{C_k(m-1)}\ , \qquad k \geq 2\ , m \geq 2\ . \tag{8}$$

Eq. (8) can be rewritten as

$$Z_k^{(s)}(m) = Z_k^{(s)}(1) + \lambda_{s-2}(m) - 1 + \sum_{j=1}^{m-1} \frac{Z_{k-1}^{(s)}(C_k(j))}{C_k(j)}\ ,$$

which, using (7), can be reduced to

$$Z_k^{(s)}(m) = \lambda_{s-2}(m) + \sum_{j=0}^{m-1} \frac{Z_{k-1}^{(s)}(C_k(j))}{C_k(j)} \qquad k \geq 2\ ,\ m \geq 1\ . \tag{9}$$

Next we derive from (9) an explicit lower bound on $Z_k^{(s)}(m)$, provided m is sufficiently large. Specifically,

Lemma 3: For each $k \geq 2$ and $m \geq C_k(k)$ a power of 2, we have

$$Z_k^{(s)}(m) \geq (k - \epsilon_k)\lambda_{s-2}(m) \tag{10}$$

where

$$\epsilon_k = \sum_{t=2}^{k} \frac{t^2-1}{C_t(t)}$$

Proof: For $k = 2$ and an arbitrary m, (6) and (9) easily imply

$$Z_2^{(s)}(m) = \lambda_{s-2}(m) + m \geq \lambda_{s-2}(m) \geq (2 - \epsilon_2)\lambda_{s-2}(m)$$

because $\epsilon_2 = 3/2$ as is easily checked.

Next suppose the claim is true up to $k-1$ for some $k > 2$, and consider the case of k and $m \geq C_k(k)$ a power of 2. Note that if $j \geq \gamma_k(m)$ then $C_k(j) \geq m \geq C_k(k) > C_{k-1}(k-1)$. Furthermore, since by definition $C_k(j)$ is a power of 2, (9) implies

$$Z_k^{(s)}(m) \geq \lambda_{s-2}(m) + \sum_{j=\gamma_k(m)}^{m-1} \frac{Z_{k-1}^{(s)}(C_k(j))}{C_k(j)} \geq$$

$$\lambda_{s-2}(m) + \sum_{j=\gamma_k(m)}^{m-1} (k-1-\epsilon_{k-1}) \cdot \frac{\lambda_{s-2}(C_k(j))}{C_k(j)} .$$

However, it is easily checked that if a divides b then $\dfrac{\lambda_{s-2}(b)}{b} \geq \dfrac{\lambda_{s-2}(a)}{a}$.

Indeed, let $b = at$. Then a feasible $DS(b,s-2)$ sequence can be obtained by concatenating t copies of a $DS(a,s-2)$ sequence of maximal length (i.e. of length $\lambda_{s-2}(a)$), composed of pairwise disjoint sets of symbols. This construction clearly proves the asserted inequality. Since in our case $C_k(j) \geq m$ for each $j \geq \gamma_k(m)$, and both $C_k(j)$ and m are powers of 2, we have

$$Z_k^{(s)}(m) \geq \lambda_{s-2}(m) + (m-\gamma_k(m)) \cdot (k-1-\epsilon_{k-1}) \cdot \frac{\lambda_{s-2}(m)}{m} =$$

$$\left[(k-\epsilon_{k-1}) - \frac{\gamma_k(m)}{m}(k-1-\epsilon_{k-1}) \right] \lambda_{s-2}(m) .$$

Next we claim that

$$\frac{\gamma_k(m)}{m} \leq \frac{k+1}{C_k(k)}$$

Indeed, note first that this inequality is immediate if $m = C_k(k)$. Assume then $\gamma_k(m) = t \geq k+1$. Then $m > C_k(t-1)$ and

$$\frac{\gamma_k(m)}{m} \leq \frac{t}{C_k(t-1)}$$

The assertion thus follows by induction from the following inequality

$$\frac{C_k(j)}{j+1} < \frac{C_k(j+1)}{j+2}$$

which in turn follows immediately from definition of C, namely $C_k(j+1) = C_k(j) \cdot C_{k-1}(C_k(j))$, and the second factor is clearly at least 2, i.e. greater than $\frac{j+2}{j+1}$.

Continuing with our inequalities, we thus have

$$Z_k^{(s)}(m) \geq \left[(k-\epsilon_{k-1}) - \frac{(k-1)(k+1)}{C_k(k)}\right]\lambda_{s-2}(m) = (k-\epsilon_k)\lambda_{s-2}(m)$$

which thus completes the inductive proof of the lemma. □

Now it is plain that the sequence ϵ_k converges to some positive limit, which we denote by ϵ. Lemma 3 thus implies

$$Z_k^{(s)}(m) \geq (k-\epsilon)\lambda_{s-2}(m)$$

for m a power of 2 greater than or equal to $C_k(k)$. We can now obtain our main result.

Theorem 4:

$$\lambda_{2s+1}(n) = \Omega(n\alpha^s(n))$$

for each $s,n \geq 1$, where the constant of proportionality depends on s.

Proof: By induction on s. The basis $s = 1$ is due to [HS]. Suppose the claim holds for $s-1$, i.e.

$$\lambda_{2s-1}(n) \geq c_{s-1} n\alpha^{s-1}(n)$$

for each $n \geq 1$. By (2) we can also write

$$\lambda_{2s-1}(n) \geq c'_{s-1} n \gamma^{s-1}(n)$$

for each $n \geq 1$ and for some appropriate positive constant c'_{s-1}. Now Lemma 3 implies

$$U_{2s+1}(k,m) \geq C_k(m)\cdot(k-\epsilon)\cdot m \cdot \frac{\lambda_{2s-1}(m)}{m} \geq c'_{s-1}(k-\epsilon)N(k,m)\gamma^{s-1}(m) .$$

Now choose $m = C_{k+1}(k)$, which clearly satisfies the assumption of the preceding lemma. Then (1) implies that $N(k,m) = C_{k+1}(k+1)$, and

$$\frac{U_{2s+1}(k,m)}{N(k,m)} \geq c'_{s-1}(k-\epsilon)\gamma^{s-1}(m) .$$

But $\gamma(m) = k+1$ because $C_k(k) < C_{k+1}(k) = m \leq C_{k+1}(k+1)$, and $\gamma(N(k,m)) = k+1$ by definition. Hence

$$U_{2s+1}(k,m) \geq c'_{s-1}(k+1)^{s-1}(k-\epsilon)N(k,m) \geq c_s N(k,m)\gamma^s(N(k,m)) .$$

for some appropriate positive constant c_s. From this we can obtain the desired lower bound for $2s+1$ and for *all* $n \geq 1$ using a straightforward argument similar to that in the proof of Corollary 6.2 of [HS]. We leave details of this argument to the reader. This completes the inductive step of the proof, and thus establishes the asserted lower bounds. □

Remark: In conclusion, we note that there still exists a considerable gap between the new lower bounds on $\lambda_s(n)$ obtained above and the upper bounds obtained in [Sh]. An obvious open problem is to narrow this gap. We believe that the actual value of $\lambda_s(n)$ is closer to the lower bounds obtained here than to the upper bounds in [Sh].

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